Tunneling of Thin Shells from Black Holes: An Ill Defined Problem

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Abstract

It is shown that $exp(-2\ Im(\int p\ dr))$ is not invariant under canonical transformations in general. Specifically for shells tunneling out of black holes, this quantity is not invariant under canonical transformations. It can be interpreted as the tunneling probability only in the cases in which it is invariant under canonical transformations. Although such cases include alpha decay, they do not include the tunneling of shells from black holes. This demonstrates that this naive expression for tunneling probability does not hold for the case of shells tunneling out of black holes.

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1 Introduction

Black holes were shown to radiate thermally by Hawking [1]. This was a result of semi-classical gravity in which field theories are quantized on classical curved spacetime backgrounds. It was also suggested by Hawking and Hartle [2] that Hawking radiation could be modeled as tunneling of particles across the horizon of the black hole. Hawking radiation was calculated for the emission of test particles (not affecting the background).

Hawking radiation poses the so called Information Paradox. Two neutral, non-rotating black holes of the same mass, formed by the collapse of completely different systems, would evaporate away leaving behind only completely identical thermal radiation. The radiation left behind by the two black holes of same mass are identical because the temperature of neutral, non-rotating black hole is a function only of its mass. In this way the information of the original collapsing systems is completely lost. For further details one can refer to [3].

One of the approaches taken to try to fix the paradox was to include the self-gravitational correction to the radiation. It was hoped that if the emitted particle's effect on spacetime curvature was also taken into account the radiation would not be thermal and the paradox would be resolved.

Israel had derived the equations of motion of self-gravitating shells [4] a decade before Hawking's derivation of Hawking radiation. However the derivation was done only for the self-gravitating shell and not shells in the presence of black holes. Additionally the results were derived from considering the Einstein's equations and not from an action. Without an action it is not clear how to quantize the shells.

An action for the self-gravitating shell was proposed by Kraus and Wilczek [5]. Parikh and Wilczek [6] worked on the idea of tunneling of shells using the action. They computed the quantity $exp(-2Im(\int p\ dr))$ where the integration domain includes the horizon (which makes the action imaginary due to a pole in the momentum). This quantity was then taken to be equal to the tunneling probability as is done for alpha particle emission. This method was then applied to several different black holes in various dimensions by many authors [7],[8],[9],[10],[11],[12],[13],[14].

The importance of [6] seemed to be that it offered a correction to Hawking radiation making it non thermal. Non thermality of the radiation was taken as a possible sign of resolution of the black hole information paradox [15],[16],[17]. It was also proposed that the non-thermality of the radiation had an effect on the inflationary vacuum [18].

Tunneling probability has to be invariant under canonical transformations. We demonstrate in this paper that the quantity $\int pdr$ is not invariant under canonical transformations in general. We show why it is invariant for *text-book* examples of tunneling and bound states (alpha decay and hydrogen atom). These reasons are not shared by the model in [6]. This makes it necessary to check the invariance of $\int p\ dr$ explicitly for this model. We demonstrate that in this case $\int p\ dr$ is not invariant under canonical transformations and hence the interpretation of $exp(-2Im(\int p\ dr))$ as the tunneling probability cannot be justified.

In this paper we work with the action for the shell due to Gladush [19]. There are several advantages of using this action over the one in [5]. Gladush was able to reproduce Israel's junction conditions [4] from the action thus lending credibility to it. The action also gives Israel's equations of motion for the self gravitating shell [4] on variation. Additionally the paper gives two canonically equivalent actions for the shell. We calculate $\int p\ dr$ from these two canonically related actions and show that the results are different from each other and from the result of [6]. Thus we conclude that $exp(-2Im(\int p\ dr))$, with the integration over the horizon, is not the correct expression for the tunneling probability of shells from black holes.

Outline of the paper

- In section 2 we show why $\int pdr$ is not invariant under canonical transformations.
- We go over the derivation of the conventional tunneling model from [6] in section 3.
- In section 4 we explain the geometry of the problem.
- We summarize the equations of motion of massless and massive shells from Israel's method [4] in section 5.
- In section 6 we derive the same equations of motion by varying Gladush's action [19].
- We calculate $exp(-2Im(\int pdr))$ in section 7 and show how the answer is different not only from the result of the conventional model [6] but also different in different canonically equivalent frames.
- In A we derive the equation of motion by Israel's method.
- In B we explain the infalling Eddington Finkelstein coordinates.
- In C alternate derivations of $exp(-2Im(\int pdr))$ are given.
- In D we motivate the action from the equations of motion.

2 Tunneling Calculations and Canonical Transformations

Quantum Mechanics allows particles to tunnel between two classically allowed regions through a classically forbidden region. The probability of such an event is given by

$$exp\left\{-2\ Im(\int_{r_{in}}^{r_{out}} p\ dr)\right\} \tag{1}$$

where the forbidden region is $r\ \epsilon\ (r_{in}, r_{out})$. However the above is not the most general case. The more generic version will be given at the end of this section.

For this quantity to be the tunneling probability it has to be invariant under canonical transformations. In general the above quantity differs in different canonically equivalent frames. If we choose a path in phase space from an initial point to a final point and calculate the action over it in two canonical frames they are related by

$$S = S' + \int_{initial}^{final} dF = S' + F_{final} - F_{initial} \tag{2}$$

where F is the generating functional for the canonical transformation and S and S' are the actions along the phase space path in the two canonically equivalent frames.

If the initial and final points are not the same in the phase space then in general we have

$$\int pdr \neq \int PdR \tag{3}$$

Specifically for tunneling purposes $\int pdr$ and $\int PdR$ may both be imaginary but unequal if F is imaginary at either the initial or final point.

So when considering tunneling do we take the probability to be

$$exp\left\{-2\ Im(\int_{initial}^{final} p\ dr)\right\} \qquad \text{or} \qquad exp\left\{-2\ Im(\int_{initial}^{final} P\ dR)\right\} \quad ?$$

There is a reason why this argument does not pose a problem in the usual tunneling models (like the alpha decay model based on tunneling). In most tunneling examples incoming and outgoing particles face the same barrier if they have the same energy or in other words their fluxes are damped by equal amounts. In such cases, as is well known (see for example Rubakov [20] [2]) the tunneling probability is given by the exponential of the negative of the action of the bounce. A bounce is a bound state of the Euclidean action. Being a bound state its action is calculated on a closed loop in the phase space and it is well known that for canonical transformations of closed loops in phase space

$$\oint pdr = \oint PdR \tag{4}$$

In other words $\int pdr$ *is not invariant under canonical transformations but* $\oint pdr$ *is.*

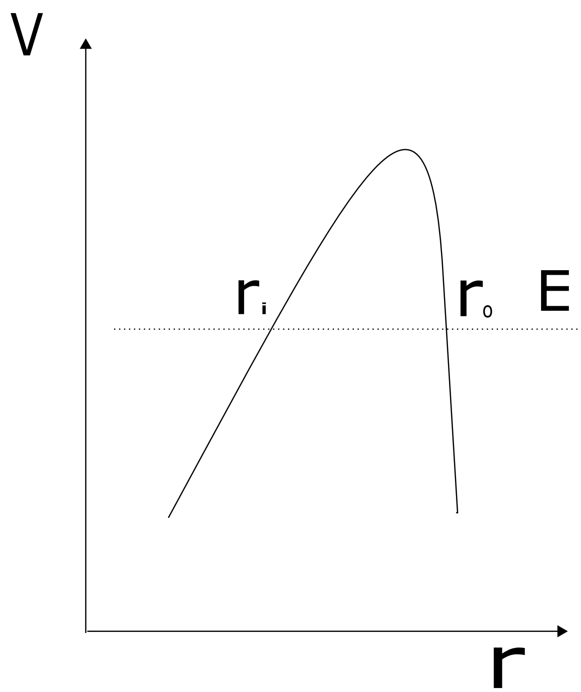

Figure 1: Regular tunneling problem

[2]This reference is a book and the relevant result is in chapter 11.

Another way to see the invariance of $exp(-2Im(\int pdr))$ for regular tunneling problems is the following. In the fig. 1 we have a barrier with potential V and a particle with energy E. In the region $r \,\epsilon\, (r_i, r_o)$ the momentum is imaginary and the tunneling probability is given as

$$
\begin{aligned}
\Gamma &\sim exp(-2\int_{r_i}^{r_o} \sqrt{2m(V-E)}dr) \\
&= exp(-\int_{r_i}^{r_o} \sqrt{2m(V-E)}dr + \int_{r_o}^{r_i} \sqrt{2m(V-E)}dr) \\
&= exp(-\oint \sqrt{2m(V-E)}dr) \\
&= exp\left\{-Im(\oint p\, dr)\right\}
\end{aligned} \tag{5}
$$

where we have taken left movers to have the opposite sign for momentum than right movers. *The integration is over a closed loop in the phase space so the result is invariant under canonical transformations*[3].

Thus we see that the correct expression for tunneling probability is

$$
exp\left\{-Im(\oint p\, dr)\right\} \tag{6}
$$

The reason that (6) could be simplified to (1) was because the left movers and right movers faced the same barrier if they had the same energy. This however is not the case for shells tunneling out of horizons in the conventional tunneling model [6]. Infalling shells face no *barrier* at all and for them $\int pdr$ does not give any imaginary part. Thus the results of tunneling probabilities in [6]-[14] are squares of what they should be.

This however is not the complete story as it will be shown in this paper that neither $exp(-2Im(\int pdr))$ nor $exp(-Im(\oint pdr))$ is invariant under canonical transformations for the case of black hole emission (where the impaginary part comes from a pole and not a forbidden region). Therefore neither of these quantities can be the probability of a black hole emitting a shell.

3 Conventional Tunneling Model: A Review

In this section the tunneling model in [6], which we refer to as the conventional model for tunneling, will be reviewed for completeness. That model used the action for shells found in [5]. Only null shells were considered. The Hamiltonian of the shell gravity system according to [5] and [6] is the ADM mass. For massless outgoing shells in Eddington Finkelstein coordinates we get the equation of motion

$$
\frac{dr}{dt} = \frac{1-\frac{2M}{r}}{2} \tag{7}
$$

[3]For bound state problems like particle in a box and hydrogen atom the same reason lets us ignore the issues of canonical transformations.

The imaginary part of $\int pdr$ was calculated by the relation

$$\int pdr = \int \int dpdr \tag{8}$$

From the relation $\dot{r} = \frac{dH}{dp}$ formally the integral was rewritten as

$$\int pdr = \int \int \frac{dH}{\dot{r}} dr \tag{9}$$

The Hamiltonian was taken to vary from M to $M - \omega$ where ω was the shell's energy. With this and (7) the value for imaginary part of $\int pdr$ was found out by integrating over the horizon and going under the pole as

$$\begin{aligned} Im(\int_{horizon} pdr) &= Im(\int_{horizon} \int_{M}^{M-\omega} \frac{dHdr}{\frac{dr}{dt}}) \\ &= Im(\int_{horizon} \int_{M}^{M-\omega} \frac{2rdHdr}{r-2H}) \\ &= \pi \int_{M}^{M-\omega} 4HdH \\ &= \pi 4\omega (M - \frac{\omega}{2}) \end{aligned} \tag{10}$$

The sign comes out to be positive if $r_{in} > r_{out}$ which was explained by saying that the horizon shrinks while emitting the shell so the tunneling process starts from just behind the horizon to emerge just outside the shrunken horizon.

Thus the tunneling probability was found to be

$$\Gamma \sim e^{-8\pi\omega M(1-\frac{\omega}{2M})} \tag{11}$$

4 Geometry and Causality

Geometry

The geometry of space time with a thin shell is non trivial and Israel [4] showed that *a singular hypersurface divides the space time that it moves in into two regions which do not share the same mass. Although it is possible to have coordinate charts which are continuous across the hypersurface for non static coordinates, it is not possible to have continuous coordinates across the hypersurface for static coordinates.* In our case the region inside/outside the shell will be referred to as $V_{\pm}$ and their mass parameters as $M_{\pm}$. The geometry is shown in fig. 2. *The horizons for both the regions are at different values of the radial parameter.*

Classically the motion of the shell can be specified completely by saying it is at $r = R(\tau)$ at $t_{-} = T_{-}(\tau)$ in terms of internal coordinates where τ is the proper time of the shell. It can also be specified completely by saying it is at $r = R(\tau)$ at $t_{+} = T_{+}(\tau)$ in terms of external coordinates. There is a canonical transformation between the internal and external coordinates [19].

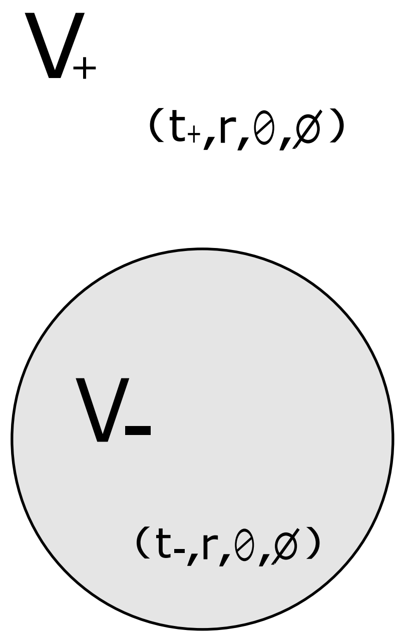

Figure 2: The geometry around a singular spherical hypersurface

The fact that the angular variables can be taken to be the same on both sides of the shell is because the geometry of the shell on both side is S^2. The radial coordinate can be taken to be the same for the same reason as it is defined as $4\pi r^2 \equiv A$ where A is the area of the S^2. The continuity of coordinates across the shell, however, cannot be maintained for the time coordinate. The presence of the shell causes the mass parameters to be different on both sides. Considering the motion of a null shell shows that *the time coordinates have to be different on both sides for static coordinates.*

In Schwarzschild coordinates the metric on both sides of the shell is

$$\begin{aligned} ds_\pm^2 &= -f_\pm dt_\pm^2 + f_\pm^{-1} dr^2 + r^2 d\Omega^2 \\ f_\pm &= 1 - \frac{2M_\pm}{r} \end{aligned} \tag{12}$$

When the shell is at a radial coordinate R, the metric on the shell, due to spherical symmetry can be taken as

$$ds_\Sigma^2 \equiv -d\tau^2 + R^2 d\Omega^2 \tag{13}$$

which defines the proper time on the shell. The relationship of the shell's proper time with the manifold coordinates is explained and equations of motion worked out in the A.

Since we have used up $\pm$ for distinguishing which manifold is being discussed we will use the symbol $\oslash$ to signify $+$ for outgoing shells and the same symbol to signify $-$ for infalling shells whenever such distinction is required (specifically for quantities calculated in Eddington-Finkelstein coordinates).

Causality

Although the shell divides spacetime into two manifolds each with a horizon, the two horizons here should not to be confused with two horizons of a two charge solution like a Reissner Nordstrom black hole.

We will refer to the horizon of the outer manifold as H_o and that of the inner manifold as H_i. Due to positivity of mass, H_o will always be at a larger radial parameter than H_i. We could have three scenarios as shown in fig. 3. The dark circle is the shell and the dotted circle is where the outer or inner horizons, as the case may be, would have been had the appropriate manifold extended to that point. The light circles are the outer and inner horizons.

The shell could be outside H_o as show in the first case. In this case the outer manifold, which reaches upto the shell, does not contain a horizon. All signals from the shell can reach the observer at infinity. The inner manifold however does have a horizon. Any signal emitted from inside H_i cannot reach the shell (and hence cannot reach the observer at infinity either).

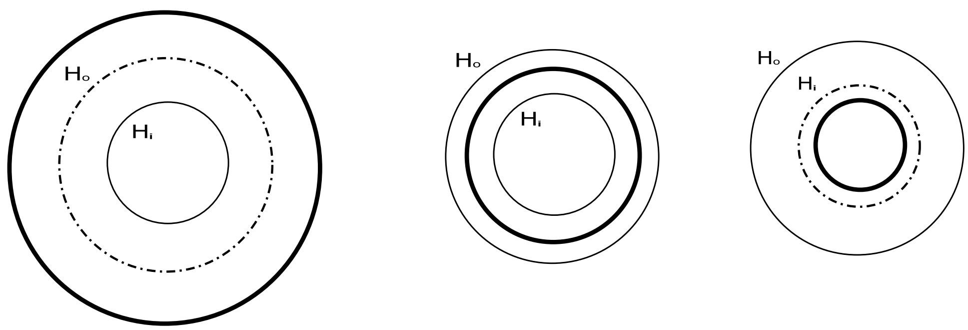

Figure 3: The dark circle shows the shell at various values of the radius parameter. The light circles show the position of the horizon. Inside the shell is V_- and outside V_+. Depending on the position and future of the shell the horizons in $V_\pm$ may or may not exist.

The second case is that the shell is between H_i and H_o. Then signals emitted from the shell cannot reach the observer at infinity. The signals emitted from inside H_i cannot reach the shell. Any outgoing signal emitted from the region between H_i and the shell would eventually emerge into the outer manifold. It will however emerge inside H_o and due to bending of the light cone will keep falling to lower values of the radial parameter. This argument also shows that once in the intermediate region the shell will eventually fall into H_i.

Finally, in the third case the shell is inside H_i. It will continue falling till it hits a singularity. Signals emitted from inside may or may not meet the shell depending on when the shell meets the singularity. If the signals meet the shell they will emerge in the outer manifold in an inward bent light cone and thus fall back to the singularity.

Thus classically once a shell crosses H_o it will meet a singularity.

5 Motion of a Thin Shell: Extending Israel's Solution

In this section the equations of motion of a thin shell moving in a Schwarzschild background are summarized. The details of the calculations can be found in A.

5.1 Motion of a Null Shell

A null shell will move along coordinates in such a way that the path length is zero. Thus the motion in outside and inside Schwarzschild coordinates from (12) is

$$(\frac{dR}{dT_\pm})^2 = F_\pm^2 \equiv f_\pm^2(R) \tag{14}$$

There is the usual coordinate singularity at the horizon of Schwarzschild coordinates (the shells seem to stop at the horizon) and we can go to some well behaved coordinate systems like Eddington-Finkelstein coordinates to study the motion through the horizon. That will be done in subsection 5.2.2 for massive shells and can be generalized to the case of null shells.

5.2 Motion of a Massive Shell

In this subsection we first summarize the equations of motion in Schwarzschild coordinates. However, we will be considering particles falling through the horizon so we then rewrite the equations of motion in infalling Eddington Finkelstein coordinates.

5.2.1 Equations of Motion in Schwarzschild Coordinates

The equations of motion are

$$\begin{aligned} \dot{T}_\pm &= \frac{\kappa_\pm}{F_\pm} \\ \dot{R}^2 &= \kappa_\pm^2 - F_\pm \\ \kappa_\pm &= \frac{1}{m}\big(\Delta M \mp \frac{m^2}{2R}\big) \end{aligned} \tag{15}$$

where m is a constant of integration which can be interpreted as the rest mass of the shell and

$$\begin{aligned} \Delta M &= M_+ - M_- \\ F_\pm &= 1 - \frac{2M_\pm}{R} \end{aligned} \tag{16}$$

We can eliminate the proper time on the shell and rewrite the equations of motion

$$\frac{dR}{dT_\pm} = \frac{F_\pm\sqrt{\kappa_\pm^2 - F_\pm}}{\kappa_\pm} \tag{17}$$

It can be seen from comparing (14) and (17) while using (15) that the limit of a null shell can be obtained by taking the rest mass of shell to vanish, $m \to 0$.

We can compare this result with the known result for test particles. We know that for test particles the equations of motion are

$$
\begin{aligned}
\dot{T} &= \frac{\kappa}{F} \\
\dot{R}^2 &= \kappa^2 - F \\
F &= 1 - \frac{2M}{R}
\end{aligned}
\tag{18}
$$

For uncharged test particles κ is a constant (and is one for test particles coming to a stop at $r = \infty$.) Thus we see that if we drop the term of order m^2 from (15) and take the particle to not influence the geometry (in other words drop the $\pm$s because we would not have two different inner and outer manifolds) we obtain (18).

We can now compare this general result with the more specific result of Israel's [4] self-gravitating shell. A self-gravitating shell is given by $M_- = 0$ and $M_+ = M$. This gives $\Delta M = M$ and the equations of motion reduce to

$$
\begin{aligned}
\dot{T}_\pm &= \frac{\frac{M}{m} \mp \frac{m}{2R}}{F_\pm} \\
\dot{R}^2 &= \Big(\frac{M}{m} \mp \frac{m}{2R}\Big)^2 - F_\pm
\end{aligned}
\tag{19}
$$

Israel specifically wrote the equation for $\dot{R}$ in terms of inner manifold quantities. In this case we have $F_- = 1$ we get the equations of motion as

$$
\begin{aligned}
\dot{T}_\pm &= \frac{\frac{M}{m} \mp \frac{m}{2R}}{F_\pm} \\
\dot{R}^2 &= \Big(\frac{M}{m} + \frac{m}{2R}\Big)^2 - 1
\end{aligned}
\tag{20}
$$

These are the same equations of motion as Israel's.

5.2.2 Equations of Motion in Infalling Eddington Finkelstein Coordinates

To study objects falling into black holes we need to go to some coordinate map which covers the future horizon. In the case of a shell we need to do this for both the manifold inside and outside the shell. In this paper we choose Eddington Finkelstein coordinates. The coordinates and transformation laws are explained in the B and here we summarize the equations of motion. As explained in section 4 we use the symbol $\oslash$ to mean $+$ for outgoing shells and $-$ for ingoing shells.

$$
\begin{aligned}
\dot{R}^2 &= \kappa_\pm^2 - F_\pm \\
\dot{T}_{\oslash,\pm} &= \frac{\kappa_\pm \oslash \dot{R}}{F_\pm}
\end{aligned}
\tag{21}
$$

Eliminating the proper time of the shell we get

$$\acute{R}_{\oslash,\pm} = \frac{dR}{dT_\pm}\Big|_{\oslash} = -(\kappa_\pm^2 - F_\pm) \oslash \kappa_\pm\sqrt{\kappa_\pm^2 - F_\pm} \tag{22}$$

We can see from (22) that the ingoing shell falls in through the horizon. The outgoing shell, however, never comes out of the horizon, which is the expected result.

6 The Action of a Thin Shell

In this section the action for the thin shell system will be discussed and the equations of motion will be derived at by varying the same. This action was proposed by Gladush [19].

6.1 The Action

It was shown in [19] that from the complete gravity action Israel's Junctions conditions can be derived. The complete gravity action gives the effective action for the shell when evaluated for the Schwarzschild solution in regions $V_\pm$. The effective action was shown to be

$$I_{sh}^{\pm} = -m\int \left(d\tau \mp U_\alpha dX^\alpha\right) \tag{23}$$

where U is a gauge potential and for Schwarzschild coordinates $U = \{-\frac{m}{2R}, 0, 0, 0\}$ for a particular gauge choice. The two actions ($\pm$) are for the same shell but in coordinates of inside or outside manifolds. It will be shown that both of them give the correct equation of motion and either should be sufficient to understand the complete motion classically. It was shown in [19] that these actions are related by canonical transformation and the explicit generating functional is given in the same.

6.2 Equations of Motion by varying the Action

We vary the action to obtain the equations of motion first in Schwarzschild coordinates and then in infalling Eddington-Finkelstein coordinates.

6.2.1 Equations of Motion in Schwarzschild Coordinates

The action (23) in Schwarzschild coordinates is

$$I_{sh}^{\pm} = -m\int (\sqrt{F_\pm - F_\pm^{-1}\acute{R}_\pm^2} \pm \frac{m}{2R})dT_\pm \tag{24}$$

where $\acute{R}_\pm \equiv \frac{dR}{dT_\pm}$. Thus the Lagrangian is

$$L_{sh}^{\pm} = -m\sqrt{F_\pm - F_\pm^{-1}\acute{R}_\pm^2} \mp \frac{m^2}{2R} \tag{25}$$

The Lagrangian is independent of the coordinate time so the Hamiltonian is a constant of motion. The conjugate momentum of the shell is

$$
\begin{aligned}
p^{\pm}_{sh} &= \frac{\partial L^{\pm}_{sh}}{\partial \acute{R}_{\pm}} \\
&= \frac{mF^{-1}_{\pm}\acute{R}_{\pm}}{\sqrt{F_{\pm} - F^{-1}_{\pm}\acute{R}^{2}_{\pm}}}
\end{aligned} \tag{26}
$$

Thus we get the Hamiltonian

$$
\begin{aligned}
H^{\pm}_{sh} &= p^{\pm}_{sh}\acute{R}_{\pm} - L^{\pm}_{sh} \\
&= \frac{mF_{\pm}}{\sqrt{F_{\pm} - F^{-1}_{\pm}\acute{R}^{2}_{\pm}}} \pm \frac{m^2}{2R}
\end{aligned} \tag{27}
$$

We can solve (27) for $\acute{R}_{\pm}$,

$$
\acute{R}_{\pm} = \frac{F_{\pm}}{H^{\pm}_{sh} \mp \frac{m^2}{2R}} \sqrt{(H^{\pm}_{sh} \mp \frac{m^2}{2R})^2 - m^2 F_{\pm}} \tag{28}
$$

By comparing (15),(17) and (28) we see that equations of motion found by Gladush's action (24) are the same as those obtained by Israel's method and the Hamiltonian is

$$
H^{\pm}_{sh} = \Delta M \tag{29}
$$

6.2.2 Equations of Motion in Infalling Eddington Finkelstein Coordinates

Here we derive the equations of motion of the shell in Eddington Finkelstein coordinates. From the laws of coordinate transformation explained in B, we observe that the gauge field in the effective action changes with the change of coordinates to $U = \{-\frac{m}{2R}, F^{-1}_{\pm}\frac{m}{2R}, 0, 0\}$. However we can gauge away the radial part because it is only a function of the radial coordinate. The action (23) in infalling Eddington Finkelstein coordinates is given by

$$
\begin{aligned}
I^{\pm}_{sh} &= -m\int (\sqrt{F_{\pm} - 2\acute{R}_{\pm}} \pm \frac{m}{2R}) dT_{\pm} \\
\acute{R}_{\pm} &\equiv \frac{dR}{dT_{\pm}}
\end{aligned} \tag{30}
$$

We calculate the conjugate momentum and the Hamiltonian which will be a constant of motion since the Lagrangian does not depend explicitly on time.

$$
\begin{aligned}
L^{\pm}_{sh} &= -m\sqrt{F_{\pm} - 2\acute{R}_{\pm}} \mp \frac{m^2}{2R} \\
p^{\pm}_{sh} &= \frac{m}{\sqrt{F_{\pm} - 2\acute{R}_{\pm}}} \\
H^{\pm}_{sh} &= \frac{m(F_{\pm} - \acute{R}_{\pm})}{\sqrt{F_{\pm} - 2\acute{R}_{\pm}}} \pm \frac{m^2}{2R}
\end{aligned} \tag{31}
$$

The last of the equations above reduces to

$$K_{\pm} = \frac{(F_{\pm} - \acute{R}_{\pm})}{\sqrt{F_{\pm} - 2\acute{R}_{\pm}}}$$

with $K_{\pm}$ defined by

$$K_{\pm} \equiv \frac{1}{m}\big(H^{\pm}_{sh} \mp \frac{m^2}{2R}\big) \tag{32}$$

We now solve for $\acute{R}_{\pm}$.

$$\acute{R}^2_{\pm} - 2\acute{R}_{\pm}(F_{\pm} - K^2_{\pm}) + F_{\pm}(F_{\pm} - K^2_{\pm}) = 0 \tag{33}$$

The roots of the equation are given by

$$\frac{dR}{dT_{\pm}}\Big|_{\oslash} = (F_{\pm} - K^2_{\pm}) \oslash K_{\pm}\sqrt{K^2_{\pm} - F_{\pm}} \tag{34}$$

Comparing (22) and(34) we see that the equation of motion obtained by varying the action is the same as that obtained by Israel's method. By comparing (15) and (32) we then get

$$\begin{aligned} K_{\pm} &= \kappa_{\pm} \\ H^{\pm}_{sh} &= \Delta M \end{aligned} \tag{35}$$

Using (31),(32) and (35) we get

$$p^{\pm}_{sh,\oslash} \;=\; m\frac{\kappa_{\pm} \oslash \sqrt{\kappa^2_{\pm} - F_{\pm}}}{F_{\pm}} \tag{36}$$

Observe that *the Hamiltonian in terms of coordinates on both sides of the shell agrees, but the momentum differs.* This will have an effect on the tunneling calculations.

7 Is $exp\left\{-2\ Im(\int p\ dr)\right\}$ the Tunneling Probability ?

Having proven that the variation of the action (23) gives the correct equations of motion in Schwarzschild as well as infalling Eddington Finkelstein coordinates, we will use the conjugate momentum from the action in infalling Eddington Finkelstein coordinates to calculate the quantity $exp(-\int pdr)$ with the integral over the future horizon. It will be shown that the answer differs when calculated in terms of internal and external coordinates.[4]

This shows that the attempt to attribute the tunneling amplitude to this quantity proves to be futile.

[4]Alternative derivations of the result are given in C to prove the robustness of the result.

7.1 Calculation of $exp\left\{-2\ Im(\int p\ dr)\right\}$

We want to calculate the quantity $exp(-2\ Im(\int p\ dr))$ with the integration over the horizon. We will work in the infalling Eddington Finkelstein coordinates since those cover the future horizon and we want to consider particles coming out of the same. From (36) and (15)

$$p^{\pm}_{sh,\oslash} \;=\; m\frac{(\frac{\Delta M}{m}R \mp \frac{m}{2}) \pm \sqrt{(\frac{\Delta M}{m}R \mp \frac{m}{2})^2 - R^2 F_{\pm}}}{(R-2M_{\pm})} \tag{37}$$

The only imaginary part of $\int pdr$, while integrating over classically forbidden regions sandwiched between classically allowed regions comes as a pole for outgoing shells. The pole comes while integrating over the future horizon. We go under the pole to get a lower probability of emission for more energetic particles. Infalling shells do not face a barrier and they do not give any imaginary action. For outgoing shells we have

$$Im(\int p\ dr) = 4\pi(M_{\pm}\Delta M \mp \frac{m^2}{4}) \tag{38}$$

Thus the exponential squared of this quantity is

$$\Gamma_{\pm} \sim= exp\left\{-2\ Im(\int p\ dr)\right\} = e^{-8\pi(M_{\pm}\Delta M \mp \frac{m^2}{4})} \tag{39}$$

This result is only valid if the shell's turning point is outside the horizon since otherwise the integration will not be over the horizon. For asymptotically free shells, $m = \Delta M$ and we get

$$\Gamma_{\pm} \sim e^{-2Im(S_{\pm})} = e^{-8\pi m(M_{\pm} \mp \frac{m}{4})} \tag{40}$$

The massless limit ($m \to 0$) which gives the correct equation of motion for a null shell reduces the above expression to

$$\Gamma_{\pm,massless} \sim e^{-8\pi M_{\pm}\Delta M} \tag{41}$$

An alternate derivation of these results which brings out the result for the massless case directly instead of by limits is given in C.

The interesting point is that the quantity that is usually associated with tunneling probability is different when calculated in the inside and outside coordinates. Mathematically this should not come as a surprise for a self gravitating shell has Minkowski vacuum inside and thus faces no barrier from inside coordinates. It does however have a horizon in the outside manifold and thus a *barrier*.

The interpretation of $\Gamma_{\pm}$ as the tunneling probability thus suffers a fatal blow.

7.2 Calculation of $exp\left\{-\ Im(\oint p\ dr)\right\}$

We had argued in section 2 that the correct expression for tunneling should be $exp\left\{-\ Im(\oint p\ dr)\right\}$ over the forbidden region. However since the infalling shells face no barrier this expression evaluates to the square root of (40) (or (41) for null shells) and is still different in different canonical frames.

So we see that this expression cannot be the probability of shells tunneling out of black holes.

8 Conclusion

We have seen in section 2 that $\Gamma \sim exp\left\{-2\ Im(\int p\ dr)\right\}$ is not invariant under general canonical transformations. We saw that this expression was a simplification of $exp\left\{-\ Im(\oint p\ dr)\right\}$. However since the infalling shells face no barrier the latter expression is the square root of the former. In this section we will refer to the the first expression as most of the literature uses that.

Since outgoing and infalling shells face different 'barriers' we proceeded to check if Γ is invariant under canonical transformations since it can be a tunneling probability only if it is invariant under canonical transformations.

After discussing Israel's equations of motion in section 5 and arriving at the same from Gladush's two canonically equivalent actions in section 6 we calculated Γ for both of the actions in section 7. We found that Γ was different in different canonically equivalent frames.

Specifically the value of Γ for shells in the two canonically equivalent frames that we used was

$$\Gamma_- \sim e^{-8\pi m(M_- + \frac{m}{4})} \tag{42}$$

in one and

$$\Gamma_+ \sim e^{-8\pi m(M_+ - \frac{m}{4})} \tag{43}$$

in the other. The two values in (42) and (43) are in general not equal.

If one were to assume that the correct expression was a geometric mean of (42) and (43) one would get $\Gamma \sim exp(-8\pi m M_{av})$ which is Hawking's result for test shells. This would suggest that there was no self interaction correction. Additionally if we we take the correct expression for $\Gamma \sim exp\left\{-\ Im(\oint p\ dr)\right\}$ we infact get the square root of Hawking's result. For massless shells we drop the second order terms in m in the above and we see that the answers still differ from each other and from the result of [6].

This shows that the naive expression for tunneling probability which works quite well for explaining alpha-decay and other semi-classical tunneling phenomenon fails to explain tunneling of shells from a black hole.

This, however, does not imply that the picture of tunneling is incorrect. It could be possible the we need a more general expression for tunneling for Hawking radiation. If and when such an expression is found it has to be tested for invariance under canonical transformations.

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A Equations of Motion from Israel's Junction Conditions

The results of Israel [4] were derived in a very elegant way by Poisson [21]. We will extend those results. The results were also extended to Reissner Nordstrom geometries[23] and we will incorporate those too. We begin with a spherical shell with surface stress energy tensor

$$S_{ab} = \sigma u_a u_b, \quad \sigma = constant \tag{44}$$

dividing space time into two Schwarzschild regions[5] $V_\pm$, with coordinates (t_+, r, θ, ϕ) and (t_-, r, θ, ϕ) and with metrics

$$\begin{aligned} ds^2_\pm &= -f_\pm dt^2_\pm + f^{-1}_\pm dr^2 + r^2 d\Omega^2 \\ f_\pm &= 1 - \frac{2M_\pm}{r} + \frac{Q^2_\pm}{r^2} \end{aligned} \tag{45}$$

When the shell is at a radial coordinate R, the metric of the shell, due to spherical symmetry can be taken as

$$ds^2_\Sigma \equiv -d\tau^2 + R^2 d\Omega^2 \tag{46}$$

The metric induced on the shell from both sides has to be same by Israel's first junction condition [4],[21] and we can set it equal to the above.

$$\begin{aligned} ds^2_{\Sigma\pm} &= -(F_\pm \dot{T}^2_\pm - F^{-1}_\pm \dot{R}^2) d\tau^2 + R^2 d\Omega^2 \\ R &= R(\tau) \quad T_\pm = T_\pm(\tau) \quad F_\pm = 1 - \frac{2M_\pm}{R} + \frac{Q^2_\pm}{R^2} \\ \dot{R} &= \frac{dR}{d\tau} \quad \dot{T}_\pm = \frac{dT_\pm}{d\tau} \end{aligned} \tag{47}$$

Here the shell is at $R(\tau)$ at the time $T_\pm(\tau)$ in the regions $V_\pm$. The requirement of the induced metric being continuous becomes

$$F_\pm \dot{T}_\pm = \sqrt{\dot{R}^2 + F_\pm} \equiv \kappa_\pm(R, \dot{R}) \tag{48}$$

The velocity of the shell particles is

$$\begin{aligned} u^\alpha_\pm &= \frac{dx^\alpha_\pm}{d\tau} \\ &= (\dot{T}_\pm, \dot{R}, 0, 0) \end{aligned} \tag{49}$$

The normal to the hypersurface formed by the word volume of the shell is gotten by the requirement

[5] By Birkhoff's theorem the geometry inside and oustide will be Reissner Nordstrom

$$n_{\pm\alpha}n^{\alpha}_{\pm} = 1 \quad n_{\pm\alpha}u^{\alpha}_{\pm} = 0 \tag{50}$$

Thus

$$n_{\pm\,\alpha} = (-\dot{R}, \dot{T}_{\pm}, 0, 0) \tag{51}$$

The extrinsic curvature on either side of the shell are defined by

$$K_{ab} \equiv n_{\alpha;\kappa}\frac{dx^{\alpha}}{dy^{a}}\frac{dx^{\kappa}}{dy^{b}} \tag{52}$$

where $\{x^{\alpha}\}$ are coordinates on the $V_{\pm}$ and $\{y^{a}\}$ are those on the shell. The angular components of K are

$$\begin{aligned} K_{\theta\theta} &= n_{\theta;\theta} = -\Gamma^{R}_{\theta\theta}n_{R} = \kappa R \\ K_{\phi\phi} &= n_{\phi;\phi} = -\Gamma^{R}_{\phi\phi}n_{R} = \kappa R sin^{2}(\theta) \end{aligned} \tag{53}$$

Now we calculate the time component of K

$$K_{\tau\tau} = n_{\alpha;\kappa}u^{\alpha}u^{\kappa} = -a^{\alpha}n_{\alpha} \tag{54}$$

on account of (50). We have acceleration perpendicular to velocity ($u_{\alpha}u^{\alpha} = \mathit{constant}$ implies $u_{\alpha}a^{\alpha} = 0$)

$$-Fa^{T}\dot{T} + F^{-1}a^{R}\dot{R} = 0 \Rightarrow a^{T} = \frac{a^{R}\dot{R}}{F^{2}\dot{T}} \tag{55}$$

Thus

$$\begin{aligned} K_{\tau\tau} &= -n_{\alpha}a^{\alpha} = \dot{R}a^{T} - \dot{T}a^{R} \\ &= -a^{R}\Big[\dot{T} - \frac{\dot{R}^{2}}{F^{2}\dot{T}}\Big] \\ &= -a^{R}\frac{\kappa^{2} - \dot{R}^{2}}{\kappa F} \\ &= -\frac{a^{R}}{\kappa} \end{aligned} \tag{56}$$

We find the acceleration in terms of other quantities

$$\begin{aligned} a^{R} &= \frac{d^{2}R}{d\tau^{2}} + \Gamma^{R}_{TT}\dot{T}^{2} + \Gamma^{R}_{RR}\dot{R}^{2} \\ &= \frac{d^{2}R}{d\tau^{2}} + \frac{1}{2}\Big[F\partial_{R}F\dot{T}^{2} + F\partial_{R}F^{-1}\dot{R}^{2}\Big] \\ &= \frac{d^{2}R}{d\tau^{2}} + \frac{1}{2}\partial_{R}F\Big[F\dot{T}^{2} - \frac{\dot{R}^{2}}{F}\Big] \\ &= \frac{d^{2}R}{d\tau^{2}} + \frac{1}{2}\partial_{R}F \end{aligned} \tag{57}$$

Also observe,

$$\frac{\dot{\kappa}}{\dot{R}} = \frac{1}{\dot{R}}\dot{R}\frac{\frac{d^2R}{d\tau^2} + \frac{1}{2}\partial_R F}{\kappa} = \frac{a^R}{\kappa} \tag{58}$$

Thus from (53),(57), (58) and (46) we get the extrinsic curvatures to be

$$\begin{aligned} K^{\tau}_{\pm\tau} &= \frac{\dot{\kappa}_{\pm}}{\dot{R}} \\ K^{\theta}_{\pm\theta} &= K^{\phi}_{\pm}\phi = \frac{\kappa_{\pm}}{R} \end{aligned} \tag{59}$$

From the second junction condition [4],[21]

$$S^a{}_b = -\frac{1}{8\pi}\big([K^a{}_b] - [K]\delta^a{}_b\big) \tag{60}$$

where $K \equiv K_{ab}h^{ab}$ and $[A] \equiv A_+ - A_-$.
Thus

$$\begin{aligned} K &= \frac{\dot{\kappa}}{\dot{R}} + 2\frac{\kappa}{R} \\ S^{\tau}_{\tau} &= \frac{[\kappa]}{4\pi R} = -\sigma \\ S^{\theta}_{\theta} &= S^{\phi}_{\phi} = \frac{[\kappa]}{8\pi R} + \frac{[\dot{\kappa}]}{\dot{R}} = 0 \end{aligned} \tag{61}$$

The solution to the second of these is

$$[\kappa]R = constant \tag{62}$$

And using this in the first

$$[\kappa]R = -\sigma 4\pi R^2 = -constant \equiv -m \tag{63}$$

Solving for $\dot{R}$ using (48) we get two versions which are equivalent

$$\dot{R}^2 \quad = \quad \frac{1}{m^2}\big[\Delta M - \frac{\Delta Q^2 \pm m^2}{2R}\big]^2 - F_{\pm} \tag{64}$$

Where $\Delta M = M_+ - M_-$ and $\Delta Q^2 = Q^2_+ - Q^2_-$. We can also get

$$\kappa_{\pm} = \frac{1}{m}\big[\Delta M - \frac{\Delta Q^2 \pm m^2}{2R}\big] \tag{65}$$

Thus the complete solution is

$$
\begin{aligned}
\kappa_{\pm} &= \frac{1}{m}[\Delta M - \frac{\Delta Q^2 \pm m^2}{2R}] \\
\dot{T}_{\pm} &= \frac{\kappa_{\pm}}{F_{\pm}} \\
\dot{R}^2 &= \kappa_{\pm}^2 - F_{\pm}
\end{aligned} \tag{66}
$$

B Eddington Finkelstein Coordinates and the Equations of Motion

The transformation law between the Schwarzschild coordinates $(t_{sc}, r_{sc}, \theta, \phi)$ and Infalling Eddington Finkelstein coordinates $(t_{ef}, r_{ef}, \theta, \phi)$ is

$$
\begin{aligned}
dr_{ef} &= dr_{sc} \\
dt_{ef} &= dt_{sc} + f^{-1} dr_{sc}
\end{aligned} \tag{67}
$$

So the radial coordinate is the same. The metric in these coordinates is

$$
ds_{\pm}^2 = -f_{\pm} dt_{ef,\pm}^2 + 2dt_{ef,\pm} dr + r^2 d\Omega^2 \tag{68}
$$

With this and the equations of motion in Schwarzschild coordinates (66) we have in Infalling Eddington Finkelstein coordinates (with $\oslash = \pm$ for outgoing and infalling shells)

$$
\begin{aligned}
\dot{T}_{ef,\oslash,\pm} &= \dot{T}_{sc,\pm} \oslash F_{\pm}^{-1} \dot{R}_{sc} \\
\dot{R}_{ef} &= \dot{R}_{sc}
\end{aligned} \tag{69}
$$

Explicitely

$$
\begin{aligned}
\kappa_{\pm} &= \frac{1}{m}[\Delta M \mp \frac{m^2}{2R}] \\
\dot{T}_{ef,\oslash,\pm} &= \frac{\kappa_{\pm} \oslash \dot{R}_{ef}}{F_{\pm}} \\
\dot{R}_{ef} &= \kappa_{\pm}^2 - F_{\pm}
\end{aligned} \tag{70}
$$

C Alternative Derivations of $exp\left\{-2\ Im(\int p\ dr)\right\}$

In section 7.1 we had evaluated the value of $exp\left\{-\ Im(\int p\ dr)\right\}$ and here we will re-derive the same by a) The Hamilton-Jacobi equation (as imaginary part of the action) and b) by using a result of black-hole emission of charged particles due to Hawking and Hartle [2].

C.1 Re-derivation using Hamilton-Jacobi Equation

This section is based on obtaining the action from the Hamilton-Jacobi equation. This method was used in [22] to arrive at the usual expression for Hawking radiation without self-interaction correction.

Massive Case
The covariant Lagrangian (23), momenta and eqn. of motion are

$$\begin{aligned}
\mathfrak{L}^{\pm} &= -m\sqrt{-\dot{x^{\mu}}\dot{x_{\mu}}} \pm U_{\mu}\dot{x}^{\mu} \\
\mathfrak{p}^{\pm}_{\mu} &= m\dot{x}_{\mu} \pm U_{\mu} \\
(\mathfrak{p}^{\pm} \mp U)^2 + m^2 &= 0
\end{aligned} \tag{71}$$

where $U = (-\frac{m^2}{2R}, 0)$.

We now apply the Hamilton-Jacobi method. Observe that $\mathfrak{L}^{\pm}$ is independent of T so we can choose the following ansatz for the action (taking the total energy the shell as from (29) and (35) as ΔM).

$$S_{\oslash,\pm} = -\Delta M T_{\pm} + W_{\oslash,\pm}(R) \tag{72}$$

Replacing $\mathfrak{p}_{\mu}$ in the equation of motion by $\partial_{\mu} S$ we get

$$-F_{\pm}(W'_{\oslash,\pm})^2 + 2(-\Delta M \pm \frac{m^2}{2R})W'_{\oslash,\pm} + m^2 = 0$$

$$W'_{\oslash,\pm} = \frac{(\Delta M R \mp \frac{m^2}{2}) \pm \sqrt{(\Delta M R \mp \frac{m^2}{2})^2 - m^2 R^2 F_{\pm}}}{R - 2M_{\pm}} \tag{73}$$

The imaginary part of the action is the same as imaginary part of $W(R)$ which is gotten by going under the pole. Hence for outgoing particles we recover

$$\Gamma_{\pm} \sim e^{-2Im(S_{\pm})} = e^{-8\pi(M_{\pm}\Delta M \mp \frac{m^2}{4})} \tag{74}$$

Massless Case
The following Hamilton Jacobi ansatz can be used for the action

$$S_{\oslash,\pm} = -ET_{\pm} + W_{\oslash,\pm}(R) \tag{75}$$

The light like shells equation of motion can be obtained by taking the massless limit of the momentum equation of massive particles

$$p^2_{\oslash,\pm} = 0 \tag{76}$$

Thus in Eddington Finkelstein coordinates we get

$$- F_{\pm} W'(R)^2_{\oslash,\pm} + 2EW'(R)_{\oslash,\pm} = 0 \tag{77}$$

The solution for the outgoing shell is

$$W(R)_{\pm} = \int dR \frac{2ER}{R - 2M_{\pm}} \tag{78}$$

The imaginary part is gotten by shifting the contour under the pole giving

$$\Gamma_{\pm} \sim e^{-2Im(S_{\pm})} = e^{-8\pi EM_{\pm}} \tag{79}$$

This matches the expression gotten by the massless limit (41).

C.2 Re-derivation using Hawking's Charged Hole Radiance Formula

In the paper [2] Hawking and Hartle argue that the probability of emission of a charged particle from a charged hole is given by

$$P_{emission} = e^{-4\pi(E - \frac{qQ}{R_H})/f'(R_H)} P_{absorption} \tag{80}$$

where q is the mass of emitted particle and Q of the RN hole and R_H is the outer horizon [6].

Although the emission probability for self energy was not given in the paper it is easy to extend the result without going into the details of the calculation. This is so because the self interaction comes as a U(1) gauge potential and the electromagnetic interaction is also a U(1) gauge interaction. So we can get the desired result by the following replacement

$$\frac{qQ}{R} \to \pm \frac{m^2}{2R} \quad \text{for } V^{\pm} \tag{81}$$

By this replacement we get the result for emission of a shell with energy ΔM as (understand here R_H would become the the horizon of whichever manifold we are considering, $R_H = 2M_{\pm}$)

$$P^{\pm}_{emission} = e^{-8\pi M_{\pm}(\Delta M \mp \frac{m^2}{4M_{\pm}})} P^{\pm}_{absorption} \tag{82}$$

Thus extending Hawking and Hartles method also gives the same result as (39) and (74).

D Meeting Newton's Laws: Motivating the Action

The derivation of equations of motion is self consistent. We did not have anything as an external force on the system. The movement is 'of its own' if you will. The question is can we have any notion of Newton's second law so that we can attribute some kind of force on the shell. Let's see what happens to the acceleration. On account of (58) and (55) we have

[6] Here outer horizon means the outer horizon of a multiple horizon black hole and not the horizon of the outer manifold of the shell. In [2] since there are only test shells, there is just one manifold.

$$a^R = \frac{\kappa_\pm \dot{\kappa}_\pm}{\dot{R}} = F_\pm \frac{\dot{\kappa}_\pm}{\dot{R}} \dot{T}_\pm = g^{RR} \frac{\dot{\kappa}_\pm}{\dot{R}} \dot{T}_\pm \tag{83}$$

$$a^T = \frac{\dot{\kappa}_\pm}{F_\pm} = -\frac{1}{F_\pm}\Big(-\frac{\dot{\kappa}_\pm}{\dot{R}}\Big) \dot{R} = -g^{TT} \frac{\dot{\kappa}_\pm}{\dot{R}} \dot{R} \tag{84}$$

we can immediately see that (83) and (84) are of the form

$$a^\mu_\pm = G^{\mu\nu}_\pm u_{\pm\,\nu} \tag{85}$$

if we identify

$$G_{\pm\;RT} = \frac{\dot{\kappa}_\pm}{\dot{R}} \tag{86}$$

If we have a κ of the form

$$\kappa_\pm = (\eta_\pm - \frac{\gamma_\pm}{R}) \tag{87}$$

for some constants $\eta_\pm$ and $\gamma_\pm$ we get

$$G_{\pm\;RT} = \frac{\gamma_\pm}{R^2} \tag{88}$$

And we see we can then explain this motion by a gauge potential A with $G = dA$ with

$$A_{\pm\;T} = -\frac{\gamma_\pm}{R} \tag{89}$$

This result matches with that in [19].

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